

Degeneracy algorithm for random magnets

S. Bastea*

*Department of Physics and Astronomy and Center for Fundamental Materials Research, Michigan State University,
East Lansing, Michigan 48824-1116*

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It has been known for a long time that the ground-state problem of random magnets, e.g., the random field Ising model, can be mapped onto the maximum flow, minimum cut problem of transportation networks. We build on this approach, relying on the concept of residual graph, and design an algorithm that we prove to be exact for finding all the minimum cuts, i.e., the ground-state degeneracy of these systems. We demonstrate that this algorithm is also relevant for the study of the ground-state properties of the dilute Ising antiferromagnet in a constant field and interfaces in random bond magnets. [S1063-651X(98)07812-X]

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I. INTRODUCTION

The statistical physics of random and frustrated systems has received a considerable deal of attention in recent years [1]. The presence of quenched disorder has been found to greatly change the bulk and interface properties of a variety of systems as compared to their “pure” counterparts, leading to different and very interesting equilibrium and non-equilibrium phenomena. Unfortunately, the progress has been many times rather slow, primarily because random systems pose sometimes insurmountable theoretical difficulties even to the most stubborn theorists. Computer simulations have played and continue to play an important role in the field, being at times the only guide through a very complicated energy landscape. While the traditional Monte Carlo method [2] proved its usefulness again and again, it was soon realized that other approaches should be considered, depending on the nature of the problem at hand. Since then a variety of algorithms, previously known only within the computer science community, have been successfully brought to bear on numerous statistical mechanics problems with quenched disorder, from spin glasses [3] to rigidity percolation [4]. Such algorithms, generally known as combinatorial optimization algorithms, have been typically used to find the exact $T=0$ ground states of the system being studied, completely avoiding the equilibration problems specific to the Monte Carlo simulations.

In the following I will focus on a single class of such algorithms, network flow algorithms, that have been put in the limelight by the work of Ogielski [5], who applied them to the study of the random field Ising model (RFIM). Since then the same method was also successfully applied to the study of equilibrium interfaces in disordered systems [6,7], becoming an important tool for the physicists working in the field. The method is generally based upon mapping the system being studied onto a network of capacities through which an incompressible fluid obeying local mass conservation flows. The problem of finding the ground state turns out to be equivalent to finding the maximum flow that can be pushed through the network between two special nodes, the

source and the sink, the so-called maximum flow, minimum cut problem of operations research [8–10]. The advantage of this approach is that polynomial time algorithms have been developed for this problem [9–11], some of which are much older than the field of random systems.

One of the problems that was usually not addressed using these algorithms and an important problem in the physics of random systems is the degeneracy of the $T=0$ ground states. In terms of the associated network flow problem this is the question of minimum cut degeneracy. An approximate algorithm dealing with this issue was proposed in [12] and applied to the RFIM problem. In this paper I build on the maximum flow, minimum cut approach, relying strongly on the concept of residual graph [11], and design an exact algorithm for finding all the minimum cuts (or, equivalently, all the ground states for a certain class of systems).

The organization of the paper is the following. For the sake of completeness Sec. II introduces the mapping of the ground-state problem to the maximum flow, minimum cut problem along with the network flow terminology and two combinatorial results that will be used in the design of the algorithm. Section III describes the degeneracy algorithm. Section IV presents a number of applications of the algorithm. Section V gives a brief conclusion. The Appendix includes the proofs to Propositions 1–4.

II. GROUND STATES USING MAXIMUM FLOW, MINIMUM CUT ALGORITHMS

In the following I will present the mapping of the ground-state problem to the maximum flow, minimum cut problem and then proceed to describe what kind of information one can extract from this mapping. I will use the $T=0$ interface problem in the random bond Ising ferromagnet to illustrate the method because it is somewhat easier for the unfamiliar reader to understand it intuitively (see also [13] for a review).

The Hamiltonian of the system is

$$H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j, \quad (1)$$

*Electronic address: bastea@pa.msu.edu

where $\sigma_i = \pm 1$ and $J_{ij} \geq 0$ are ferromagnetic couplings between neighboring spins. J_{ij} are fixed independent identically distributed random variables: quenched randomness. If the system is a d -dimensional cube, an interface with dimension $d-1$ can be induced by using periodic boundary conditions along the $d-1$ directions and setting the spins in the two $(d-1)$ -dimensional hyperplanes that represent the boundaries of the system along the last direction to $+1$ and -1 , respectively. The interface that will form in the system between the $+1$ and -1 hyperplanes will generally be rough because it will wander in order to break the weakest bonds. The energy is a minimum over all the possible spin configurations (with the ± 1 boundary spins fixed) and therefore the problem of finding the minimum energy configuration(s) would appear to be computationally very hard, even for small system sizes. As it turns out, this is not the case if one takes advantage of the similarity between this problem and the maximum flow, minimum cut problem, very well known in the study of transportation networks [8–10]. The idea is the following. Two new extra sites are introduced, a source node s that is connected to all the spins of the $+1$ hyperplane and a sink node t connected to all the spins of the -1 hyperplane. The ferromagnetic constants coupling the source node σ_s and the sink node σ_t to their corresponding hyperplane are chosen to be strong enough so they are not broken in the ground state. Then, by setting $\sigma_s = +1$ and $\sigma_t = -1$ an interface is induced as before. Now we view the system, including s and t , as a graph whose arcs are the bonds between the spins. The arcs have forward and backward capacities equal to the corresponding coupling constants $c_{ij} = c_{ji} = J_{ij}$, or we can imagine that the nodes i and j are connected by both forward and backward arcs with capacities c_{ij} and c_{ji} , so this is a directed graph. (The constraint that $c_{ij} = c_{ji}$ is dictated by the physics and is not specific to the general maximum flow, minimum cut problem. It can be relaxed for the interface problem, but not for the random field problem.) We define the set of nodes as N and a partition (X, Y) of the nodes as

$$X \equiv \{i \in N | \sigma_i = +1\}, \quad (2)$$

$$Y \equiv \{i \in N | \sigma_i = -1\}. \quad (3)$$

Then $X \cup Y = N$, $X \cap Y = \emptyset$, $s \in X$, and $t \in Y$. The knowledge of such a partition determines the energy of the spin configuration and the position of the interface. This is readily seen if we write the Hamiltonian of the system as

$$\begin{aligned} H &= - \sum_{(i,j) \in A(X)} J_{ij} - \sum_{(i,j) \in A(Y)} J_{ij} + \sum_{(i,j) \in A(X,Y)} J_{ij} \\ &= H_0 + 2 \sum_{(i,j) \in A(X,Y)} J_{ij}, \end{aligned} \quad (4)$$

where H_0 is the energy of the fully aligned system, $H_0 = \sum_{(i,j) \in A} J_{ij}$, and we defined $A(X) \equiv \{(i,j) | i \in X, j \in X\}$, $A(Y) \equiv \{(i,j) | i \in Y, j \in Y\}$, and $A(X,Y) \equiv \{(i,j) | i \in X, j \in Y\}$. Thus the problem of finding the ground-state interface, which has minimum energy, is equivalent to finding a partition (X, Y) , also called a cut, that minimizes $\sum_{(i,j) \in A(X,Y)} J_{ij}$, *minimum cut*. (Note that H_0 is a constant for a given random sample.) If we imagine fluid flowing through the network

from the source s to the sink t , the minimum cuts are the bottlenecks of the network of capacities because they determine the maximum flow that can be pushed through the network from s to t . It is useful at this point to put the problem in a more general setting. For this a number of definitions are in order.

A *transportation network* is a directed graph $G_c(N, A)$ with two special nodes: s the source and t the sink or target; N is the set of nodes and A the set of arcs. The directed arc (ij) going from node i to j has capacity $c_{ij} \geq 0$. [For the sake of clarity we assume that if the arc (ij) exists and has capacity $c_{ij} > 0$ then (ji) also exists and has capacity $c_{ji} \geq 0$.] A *flow* through the network $G_c(N, A)$ is a set of numbers $\{f_{ij}\}$, each corresponding to an arc in A , subject to the feasibility constraints

$$0 \leq f_{ij} \leq c_{ij} \quad (\text{capacity constraint}), \quad (5)$$

$$\begin{aligned} e_i &= \sum_{\{j | (ji) \in A\}} f_{ji} - \sum_{\{j | (ij) \in A\}} f_{ij} = 0 \\ & \quad (\text{local flow conservation}) \end{aligned} \quad (6)$$

for all the nodes $j \in N - \{s, t\}$ and

$$-e_s = e_t = f, \quad (7)$$

where f is called the value of the flow. (Note that $-e_s = e_t$ follows from the flow conservation [9].) The *maximum flow problem* of network flows is concerned with finding the flow $\{f_{ij}\}$ through the transportation network G_c that has maximum value f .

For any feasible flow through the network we define the *residual graph* as the graph $G_f(N, A)$ with positive *residual capacities* of the arcs,

$$r_{ij} = c_{ij} - f_{ij} + f_{ji} > 0, \quad (8)$$

where $f_{ij} - f_{ji}$ is the net flow from i to j . (Note that $r_{ij} \geq 0$ follows from the capacity constraint and also that it is possible that $c_{ij} = 0$ and $r_{ij} > 0$ when $c_{ji} > 0$.) An *augmenting path* is a directed path from s to t in the residual graph G_f .

A *cut* is a partition of the nodes set N into two subsets X and Y , denoted by (X, Y) , with $s \in X$ and $t \in Y$, such that $X \cup Y = N$ and $X \cap Y = \emptyset$. The *capacity of a cut* is defined as

$$c(X, Y) = \sum_{\{(ij) \in A | i \in X, j \in Y\}} c_{ij}. \quad (9)$$

In the following I will concentrate on the case of a transportation network $G_c(N, A)$ in which if the arc (ij) exists and has capacity $c_{ij} > 0$ then the arc (ji) also exists and has capacity $c_{ji} > 0$. This kind of network is the most relevant one in the physics applications that I described before. For these networks the following two propositions can be proved (the proofs are contained in the Appendix), which can be used to design an algorithm that finds all the minimum cuts in the network $G_c(N, A)$.

Proposition 1. If $\{f_{ij}\}$ is a maximal flow and (X, Y) a minimum cut then $r_{ij} = 0$ for all arcs $\{(ij) \in A | i \in X, j \in Y\}$.

Proposition 2. If $f_{max} > 0$ a cut (X, Y) in G_c is a minimum cut if and only if it is a directed partition in G_f^{max} , the residual graph for a maximal flow.

The calculation of an actual maximal flow through the network can be done using polynomial time algorithms. The first such algorithm, the *augmenting path algorithm* [9], was proposed originally by Ford and Fulkerson and it is also a way to prove the maximum flow, minimum cut theorem (see the Appendix). However, much faster algorithms have been developed in recent years, in particular push-relabel methods with global updates, that allow one to deal with much bigger systems than before [10]. These algorithms can be and have been used in such a way as to improve the computational speed by providing only the value of the maximal flow and a minimum cut, but not an actual maximal flow through the network [11]. However, the knowledge of an actual maximal

flow through the network is crucial for finding all the minimum cuts. Fortunately, such a calculation can be made without a major loss of speed [11].

III. DEGENERACY ALGORITHM

In the following I will use Propositions 1 and 2 to design an algorithm that finds all the minimum cuts. The algorithm will be aimed at finding the set of arcs $\{(ij) \in A | i \in X, j \in Y \text{ for all minimum cuts } (X, Y)\}$, denoted hereinafter by \mathcal{A}_{mc} , and a procedure for determining the actual minimum cuts.

Let us assume that we constructed a maximal flow $\{f_{ij}^{max}\}$ through the network $G_c(N, A)$ using an appropriate algorithm and let G_f^{max} be the associated residual graph. We have the definition

$$Y_t \equiv \{\text{all nodes that can reach the sink along a directed path in } G_f^{max}\}, \quad (10)$$

$$X_t = N - Y_t. \quad (11)$$

Then (X_t, Y_t) is a minimum cut that we will call the T cut (this follows from Proposition 2). This minimum cut has the property that for any other minimum cut (X, Y) , $Y \cap Y_t = Y_t \neq Y_t$; then $X_1 = Y_t - Y_1 \subset X$. Because (X, Y) is a directed cut (Proposition 2), for any node $i \in X$ there is no directed path from i to the sink t in G_f^{max} , which must also be true for any $i \in X_1$. However, $X_1 \subset Y_t$, so this is a contradiction.] We have also the definition

$$X_s \equiv \{\text{all nodes that can be reached from the source along a directed path in } G_f^{max}\}, \quad (12)$$

$$Y_s = N - X_s. \quad (13)$$

Then (X_s, Y_s) is also a minimum cut that we call the S cut. This minimum cut has the property that for any other minimum cut (X, Y) , $X \cap X_s = X_s$. Now we define

$$Z = N - X_s - Y_t. \quad (14)$$

Then if $Z = \emptyset$ a single minimum cut exists; otherwise there are at least two. We will be concerned with the nontrivial case $Z \neq \emptyset$.

At this point our knowledge of \mathcal{A}_{mc} is summarized in Fig. 1. The arrows stand for possibly more than one arc of $G(N, A)$ and all these arcs are included in \mathcal{A}_{mc} . [Note that these arcs are saturated by the flow, that is $r_{ij} = 0$, so in G_f^{max} only the arcs (ji) are present.] By the construction of Z we also know that all the remaining arcs that make up \mathcal{A}_{mc} connect nodes that are included in Z .

The first part of the algorithm is aimed at finding the disconnected pieces (clusters) that make up Z and is therefore called *cluster counting*. A look at Fig. 4, which is an application of the algorithm to the bimodal RFIM, clarifies the significance of this step. The idea is to determine the connectivity of Z in G_f^{max} using as connectivity rule ($r_{ij} \neq 0$ or $r_{ji} \neq 0$), which is equivalent to ($c_{ij} \neq 0$ or $c_{ji} \neq 0$.) This procedure does not reveal any new arcs of \mathcal{A}_{mc} , but shows how the minimum cuts are constructed using the arcs of \mathcal{A}_{mc} that we know at this moment. The well known Hoshen-Kopelman algorithm [14], which is an efficient clus-

ter labeling procedure, can be readily adapted for the cluster counting task and allows such a calculation to be made in a time proportional to the number of nodes in Z [15]. Figure 2 summarizes our knowledge of the minimum cuts after this step. It is easy to see that the number of minimum cuts that we can construct at this time is $2^{n(Z)}$, where $n(Z)$ is the number of independent clusters making up Z . Our search for the remaining minimum cuts is then reduced to finding the

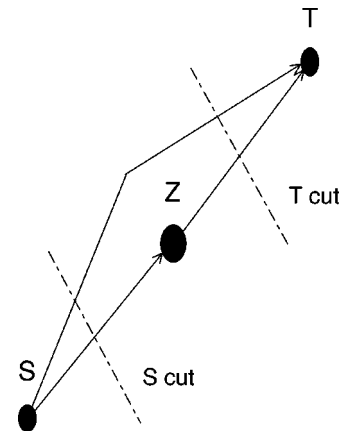


FIG. 1. Simplified representation of the network after the first step of the algorithm, showing the S cut, the T cut, and Z (we denote X_s by S and Y_t by T ; see text). Note that a directed arc connecting S and T may or may not exist.

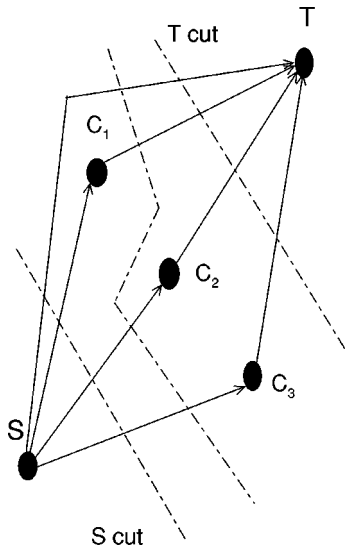


FIG. 2. Simplified representation of the network after the clusters making up Z have been identified and an example of a minimum cut.

minimum cuts in each of the independent clusters. If $f_{max} = 0$ the algorithm can be stopped here as everything is known about the minimum cuts. Any partition of the $n(Z)$ clusters is a minimum cut and the total number of minimum cuts is $2^{n(Z)}$. (Note that if $f_{max} > 0$ the “zero” step of the algorithm should be to find the cluster percolating from s to t with the above connectivity rule. All the other clusters contribute in a trivial manner to the minimum cuts; therefore, in the following we assume that they have been already discarded.)

In the second part of the algorithm we perform a *subcluster counting* procedure on each of the independent clusters

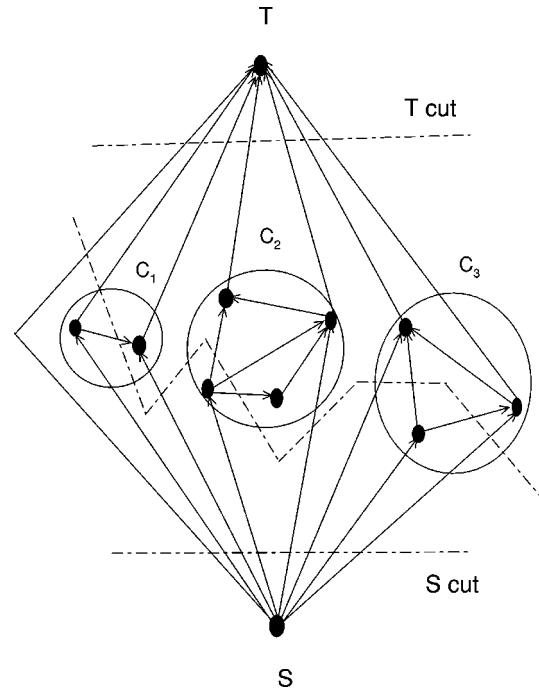


FIG. 3. Supergraph $\mathcal{G}(N, A)$ at the end of the algorithm and an example of a minimum cut.

found at the first step. The connectivity rule that we use is ($r_{ij} \neq 0$ and $r_{ji} \neq 0$.) It is clear, using Proposition 1, that the subclusters so obtained can only be on one side or the other of a minimum cut, i.e., the set \mathcal{A}_{mc} is included in the set of arcs that connect these subclusters to one another (we formally call X_s and Y_t subclusters). In order to eliminate the overcounting, we apply a third procedure called *subcluster coagulation*.

At the end of the second step of the algorithm the sub-

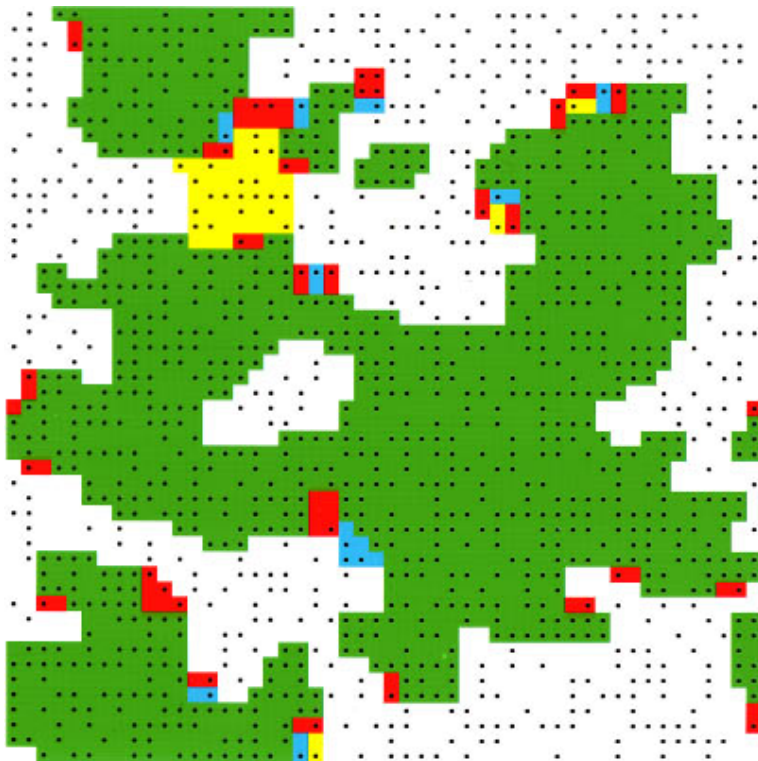


FIG. 4. (Color) Ground-state structure of the bimodal RFIM for $h/J=3/2$. The spins frozen “up” are green, the spins frozen “down” are white, and the other colors represent the spins making up Z ; neighboring subclusters have different colors (see the text). Note that Z is made up of independent clusters that contain one or more subclusters. A dot indicates a field “up” and the absence of it indicates a field “down.”

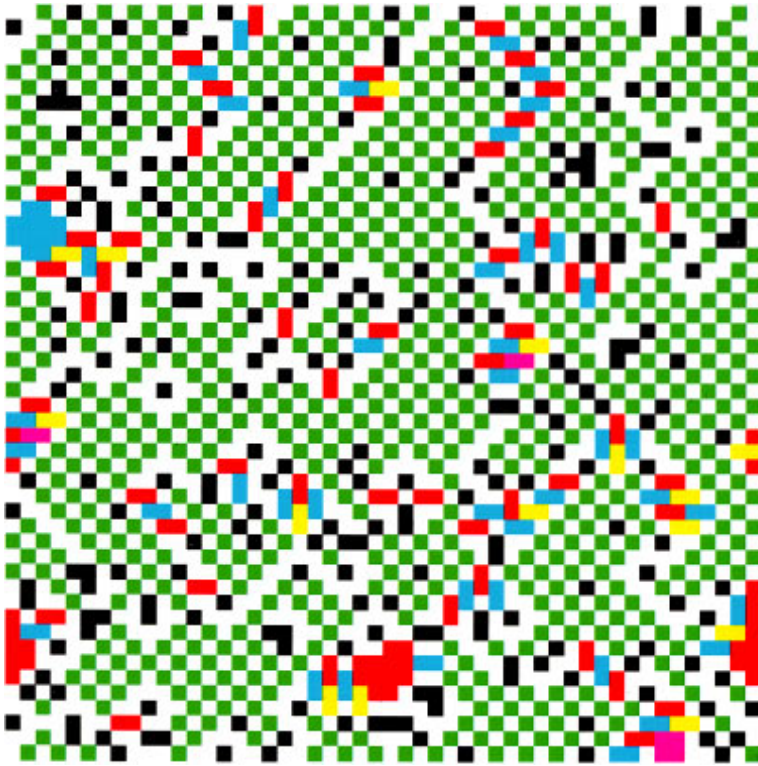


FIG. 5. (Color) Ground-state structure of the DAFF for $h/J=7/2$ and $p=0.9$. The color coding is the same as for the RFIM, only now black indicates an empty (“nonmagnetic”) site. Note that in this case “flipping” a (sub)cluster means coloring it with one or the other of the checkered patterns.

clusters are connected with each other possibly through multiple arcs with $r_{ij}=0$ and $r_{ji}>0$. The *subcluster coagulation* procedure is applied iteratively on each of the independent clusters and consists of the coagulation of subclusters that make up a directed cycle. A directed cycle is an ordered sequence of subclusters S_1, S_2, \dots, S_m , such that for all $S_k, k=1, \dots, m, i_k, j_k$ nodes exist, $i_k, j_k \in S_k$, not necessarily all distinct, with the property that $r_{i_k j_{k+1}}=0$ and $r_{j_{k+1} i_k}>0$, where $S_{m+1} \equiv S_1$. The idea is that, according to Proposition 2, subclusters making up such a directed cycle cannot be on different parts of a minimum cut; therefore, the arcs connecting them are not included in \mathcal{A}_{mc} (see also the discussion following Proposition 4). A Hoshen-Kopelman-type algorithm, in which the subcluster coagulation is achieved through relabeling of the subclusters, is again rather efficient at handling this task, but managing the data structure requires rather intricate coding.

At the end of the algorithm we have constructed a supergraph like the one in Fig. 3, which we denote by $\mathcal{G}(\mathcal{N}, \mathcal{A})$, with single directed arcs and no cycles. The nodes are now the subclusters and the arcs stand, as before, for possibly more than one arc of $G(N, A)$. Formally, the arc (IJ) going from the subcluster I to subcluster J is defined by

$$(IJ) = \{(ij) \in A \mid i \in I, j \in J\} \quad (15)$$

and by the construction of the algorithm $r_{ij}=0$ and $r_{ji}>0$ for all arcs $(ij) \in (IJ)$. Then we define $\mathcal{A}_{sg} = \cup_{\mathcal{A}} (IJ)$, the set of arcs of $G(N, A)$ represented by the arcs of $\mathcal{G}(\mathcal{N}, \mathcal{A})$. The following proposition is then true (see the Appendix for the proof).

Proposition 3. At the end of the algorithm $\mathcal{A}_{sg} = \mathcal{A}_{mc}$.

When the algorithm terminates \mathcal{A}_{mc} is known and, moreover, the problem of counting the minimum cuts is reduced to finding all the directed partitions in a directed, much

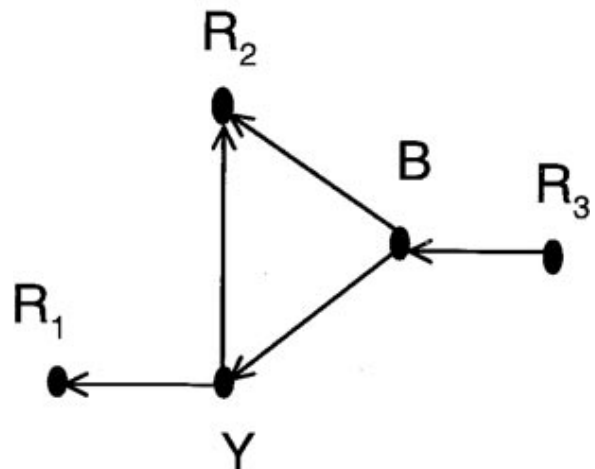
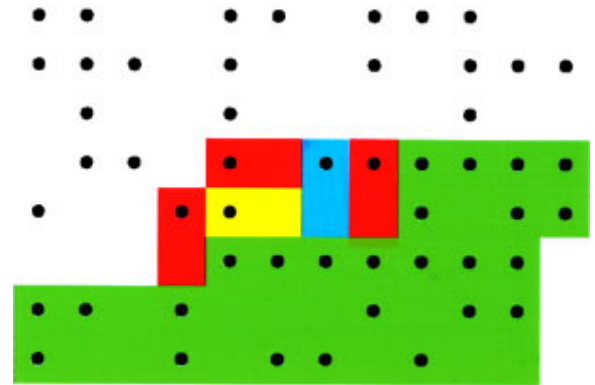


FIG. 6. (Color) RFIM cluster from Fig. 4 and its subcluster graph representation; R stands for red, Y for yellow, and B for blue. The associated degeneracy is 7.

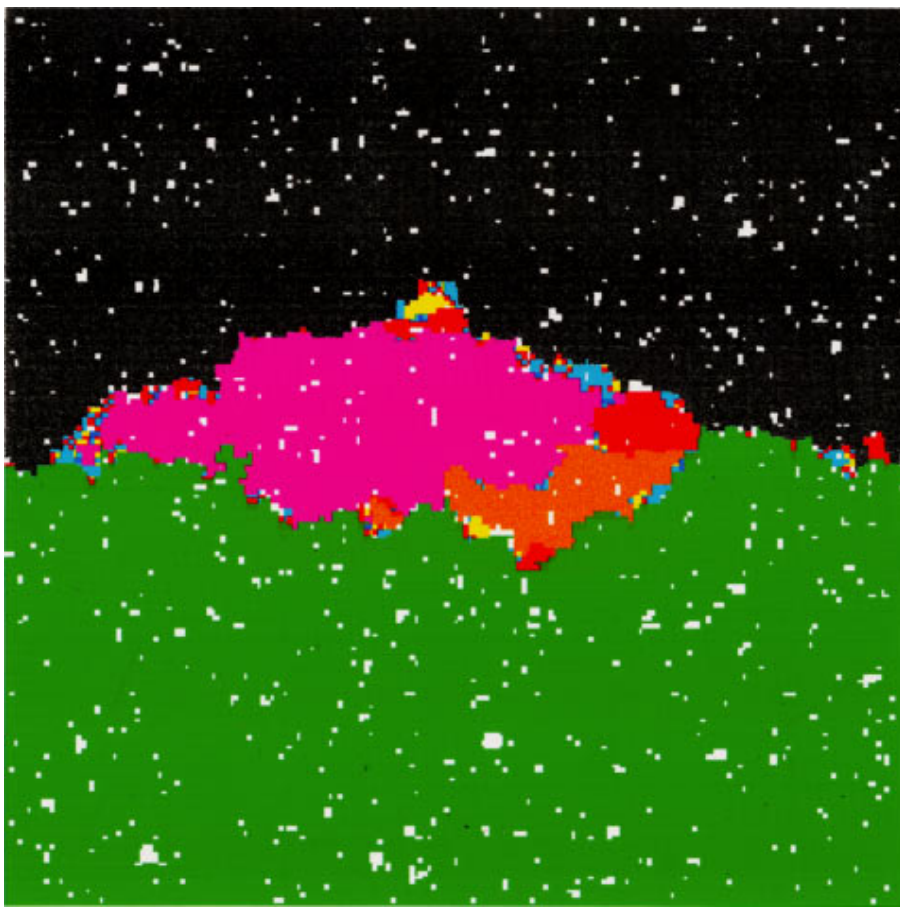


FIG. 7. (Color) Interface configuration for the bond-diluted Ising model at $p=0.64$. The spins frozen up are green, the spins frozen down are gray, and the pieces that are not part of the percolating cluster (see the text) are white. The other colors represent again the spins making up Z , with neighboring subclusters having different colors. Note that for this particular sample Z is made up of six independent clusters.

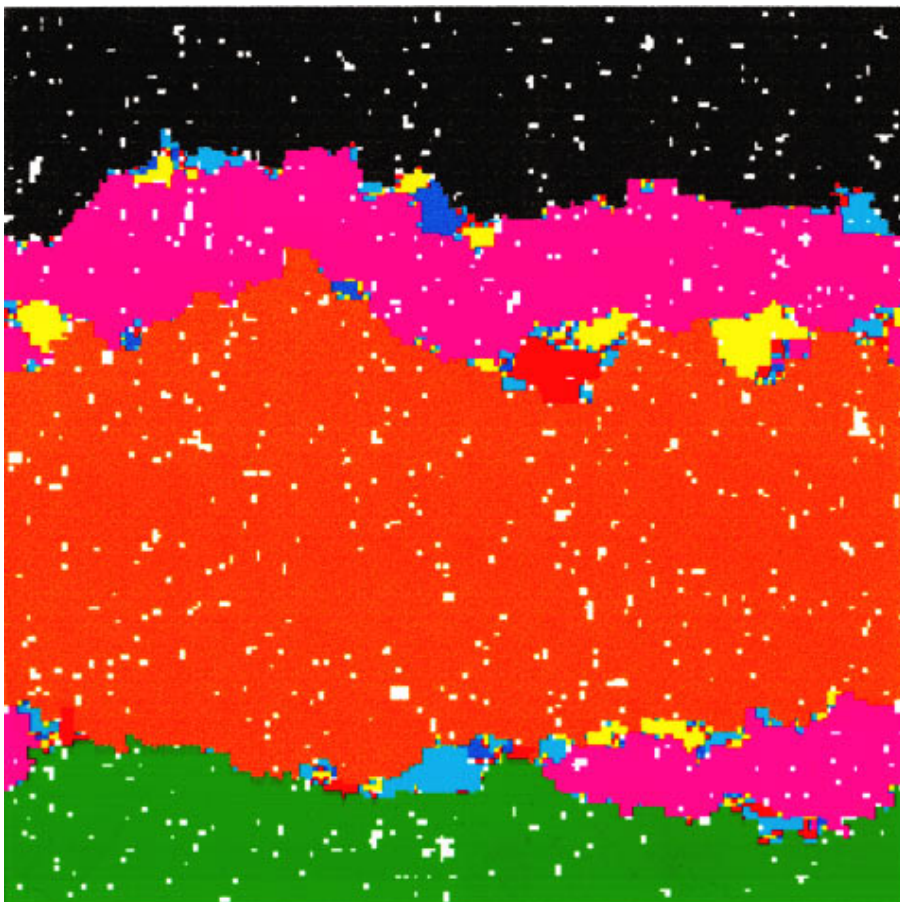


FIG. 8. (Color) Different sample from Fig. 7, which has three independent interfaces, each with its own excitations. Note that Z is in this case a single cluster in the algorithm denomination.

smaller supergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$ with single arcs and no cycles. Simple enumeration is therefore feasible if the independent clusters are not too big. The degeneracy (total number of minimum cuts) D can be written as

$$D = \prod_i d(i), \quad (16)$$

where $d(i)$ is the degeneracy associated with cluster i , $d(i) \geq 2$.

One additional result can be proved (the proof is contained in the Appendix), which was also probably known by Ford and Fulkerson [8], that further clarifies the meaning of \mathcal{A}_{mc} .

Proposition 4. \mathcal{A}_{mc} is the set of arcs of $G(N, \mathcal{A})$ that will be saturated by all the maximal flows (the ‘‘weak’’ links of the network).

The above proposition makes it easier to understand intuitively the significance of the subcluster coagulation step of the algorithm. It is clear that the flows between subclusters are all saturating and also that the flow conservation holds for each subcluster as a unit. Therefore, if for a certain maximal flow we identify a saturating flow cycle between subclusters, we can reduce the flow around the cycle by its smallest value between two subclusters making up the cycle, while keeping the overall flow maximal. This implies that none of the arcs making up the cycle is necessarily saturated when the flow is maximal; therefore, according to Proposition 4, they are not part of \mathcal{A}_{mc} . Furthermore, because of the way the subclusters have been constructed, none of the arcs connecting the subclusters making up the cycle is part of \mathcal{A}_{mc} , so the subclusters can be coagulated.

IV. APPLICATIONS

Our interest in designing a degeneracy algorithm arose in connection with our desire to study the ground-state properties of random magnets. We first used the above algorithm to study the ground-state structure of the two-dimensional $\pm h$ RFIM, which we expected to have a large degeneracy. The Hamiltonian of the RFIM is

$$H_{RFIM} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i. \quad (17)$$

$J > 0$ and the h_i 's are independent random variables drawn from a symmetric distribution $P(h_i)$, with $\langle h_i \rangle = 0$ and $\langle h_i^2 \rangle^{1/2} = h$. In the case of the $\pm h$ RFIM the h_i 's are, with equal probability, $+h$ and $-h$. This problem is mapped onto the network flow problem by connecting the sites with positive fields to the source and the ones with negative fields to the sink through arcs with capacity h (see, for example, [13] for more details).

The structure of a $\pm h$ RFIM ground state is shown in Fig. 4: two frozen ferromagnetic domains of ‘‘up’’ and ‘‘down’’ spins and a number of isolated clusters that can be flipped (colored) independently of each other to generate new ground states. There is also a degeneracy associated with flipping certain groups of subclusters inside a cluster. Surprisingly, domains that can be flipped without changing the energy exist even if h/J is irrational. In this case they have

zero field energy and the same exchange energy in the ‘‘up’’ and ‘‘down’’ states and are located at the boundary of the frozen domains. The smallest such clusters have only two spins (see Fig. 4). As a result of this structure the $\pm h$ RFIM has a strictly positive entropy for a range of h/J and related with it a new order parameter, the paramagnetic response associated with the orientation of the above domains. This result may be relevant to universality issues that are currently being debated [16,17].

A similar structure is found for the dilute Ising antiferromagnet in a constant field (DAFF) (Fig. 5), which is believed to be the experimental realization of the RFIM:

$$H_{DAFF} = J \sum_{\langle ij \rangle} \epsilon_i \epsilon_j \sigma_i \sigma_j - h \sum_i \epsilon_i \sigma_i. \quad (18)$$

$J > 0$ and ϵ_i are quenched independent random variables and $P(\epsilon_i) = p \delta(\epsilon_i - 1) + (1-p) \delta(\epsilon_i)$, $0 < p \leq 1$. Detailed results are reported elsewhere [18].

For the sake of clarity one of the clusters in Fig. 4 and its subcluster graphic representation as given by the algorithm are shown in Fig. 6. The connections to X_s and Y_t are not shown; any directed partition is a minimum cut. Note that the arrows separating such a partition point from the up subclusters to the down subclusters.

We also applied the algorithm to the study of the ground-state interfaces in the bond-diluted Ising model. The Hamiltonian is the one in Sec. II, with J_{ij} being $J > 0$ with probability p and 0 with probability $1-p$. The ground-state structure is shown in Fig. 7, with neighboring subclusters having different colors. These subclusters can be thought of as the excitations of a single interface, as in Fig. 7, or multiple interfaces, as in Fig. 8. The size distribution of these excitations is a power law. Detailed results will be reported elsewhere [19].

V. CONCLUSION

In conclusion, we designed an exact algorithm that finds all the arcs of a flow network that are part of a minimum cut and allows the effective construction of all the minimum cuts. This algorithm is relevant for the study of the ground-state properties of the random field Ising model, dilute Ising antiferromagnet in a field, interface properties of certain Ising models with bond or site disorder, and other physical problems that can be mapped onto network flow problems and where the ground state is expected to be degenerate.

After this work was completed Bruce Hendrickson brought to my attention the work of Ball and Provan [20], which contains the idea of constructing an acyclic graph that can be viewed as a compact representation of all minimum cuts (see also [21]). The authors also present a polynomial algorithm for counting the directed cuts in a planar acyclic graph. It appears that these problems are of continuing interest for the study of network reliability.

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APPENDIX

In the following we present the proofs of the Propositions 1–4 that were quoted in the main text. They rely in part on the maximum flow, minimum cut theorem of Ford and Fulkerson [8,9], probably the most important result in network flows.

Ford-Fulkerson Theorem. In a transportation network $G_c(N,A)$ the maximum value of f over all flows $\{f_{ij}\}$ is equal to the minimum value $c(X,Y)$ over all cuts (X,Y) .

Proposition 1. If $\{f_{ij}\}$ is a maximal flow and (X,Y) a minimum cut then $r_{ij}=0$ for all arcs $\{(ij) \in A | i \in X, j \in Y\}$.

Proof. Let $\{f_{ij}\}$ be a maximal flow with value f_{max} and (X,Y) a minimum cut. We have $f=f_{max}=\sum_{\{(ij) \in A | i \in X, j \in Y\}}(f_{ij}-f_{ji})$ by conservation of the flow [9] and also $f_{max}=\sum_{\{(ij) \in A | i \in X, j \in Y\}}c_{ij}$ by the Ford-Fulkerson theorem. This immediately implies $\sum_{\{(ij) \in A | i \in X, j \in Y\}}r_{ij}=0$ and because $r_{ij} \geq 0$, Proposition 1 follows.

Proposition 2. If $f_{max} > 0$ a cut (X,Y) in G_c is a minimum cut if and only if it is a directed partition in G_f^{max} , the residual graph for a maximal flow.

Definition. A partition (X^*,Y^*) of the nodes N , $X^* \cup Y^* = N$, $X^* \cap Y^* = \emptyset$, is directed if the arcs connecting X^* and Y^* all have the same direction, for example, going from Y^* to X^* , i.e., $r_{ij}=0$ for all arcs $\{(ij) \in A | i \in X^*, j \in Y^*\}$ and $\exists(j^*i^*) \in A$, $i^* \in X^*$, $j^* \in Y^*$, $r_{j^*i^*} > 0$.

Proof. If (X,Y) is a minimum cut we have from Proposition 1 $r_{ij}=0$ for all arcs $\{(ij) \in A | i \in X, j \in Y\}$. Now if we also assume $r_{ji}=0$ for all arcs $\{(ji) \in A | i \in X, j \in Y\}$ this implies $c_{ij}=c_{ji}=0$ for all arcs $\{(ij) \in A, (ji) \in A | i \in X, j \in Y\}$ and further that $f_{max}=0$ from the Ford-Fulkerson theorem, which is a contradiction. Thus the direct implication is proved.

Conversely, let us now assume that (X^*,Y^*) is a directed partition in G_f^{max} , i.e., $r_{ij}=0$ for all arcs $\{(ij) \in A | i \in X^*, j \in Y^*\}$ and $\exists(j^*i^*) \in A$, $i^* \in X^*$, $j^* \in Y^*$, $r_{j^*i^*} > 0$. In or-

der to prove that (X^*,Y^*) is a minimum cut we have to prove first that (X^*,Y^*) is a cut, i.e. $s \in X^*$ and $t \in Y^*$. Let us assume that this is not true. Then either (a) $s \in Y^*$, $t \in X^*$, (b) $s \in X^*$, $t \in X^*$, or (c) $s \in Y^*$, $t \in Y^*$. We will show that all these lead to a contradiction.

(a) $s \in Y^*$, $t \in X^*$. In this case (Y^*,X^*) is a cut and by the conservation of the flow $\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}(f_{ji}-f_{ij})=f_{max} > 0$. However, $r_{ij}=0$ implies $f_{ji}-f_{ij}=-c_{ij}$, so $\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}(f_{ji}-f_{ij})=-\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}c_{ij} \leq 0$, and therefore is a contradiction. (In fact, the inequality is strict, $\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}c_{ij} > 0$, because $f_{max} > 0$, and c_{ij} and c_{ji} are simultaneously zero or strictly positive.)

(b) $s \in X^*$, $t \in X^*$. In this case the flow conservation implies that the net flow into Y^* must be zero, i.e., $\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}(f_{ij}-f_{ji})=0$. Therefore, $\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}r_{ij}=\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}c_{ij}=0$ and $\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}r_{ji}=\sum_{\{(ij) \in A | i \in X^*, j \in Y^*\}}c_{ji} > 0$. As a result $\exists(ij) \in A$ with $c_{ij}=0$ and $c_{ji} > 0$, which is a contradiction. Case (c) is similar to (b). From all the above it follows that $s \in X^*$ and $t \in Y^*$, so (X^*,Y^*) is a cut.

Now we have to prove that (X^*,Y^*) is also a minimum cut. We have $\sum_{\{(ij) \in A | i \in X, j \in Y\}}r_{ij}=0$ and therefore

$$\begin{aligned} c(X,Y) &= \sum_{\{(ij) \in A | i \in X, j \in Y\}} c_{ij} \\ &= \sum_{\{(ij) \in A | i \in X, j \in Y\}} (f_{ij}-f_{ji}) = f_{max} \end{aligned} \quad (A1)$$

by the conservation of the flow. This implies, according to the Ford-Fulkerson theorem, that (X,Y) is a minimum cut.

Proposition 3. At the end of the algorithm $\mathcal{A}_{sg} = \mathcal{A}_{mc}$.

Proof. $\mathcal{A}_{mc} \subset \mathcal{A}_{sg}$ follows from the construction of the algorithm and Propositions 1 and 2. In order to prove $\mathcal{A}_{sg} \subset \mathcal{A}_{mc}$ let us assume, without any loss of generality, that the supergraph has a single cluster (multiple clusters are independent of each other). Let (IJ) be an arc of $\mathcal{G}(\mathcal{N},\mathcal{A})$ connecting subclusters I and J . (We consider only the non-trivial case $I \neq Y_t$ and $J \neq X_s$; if $I = Y_t$ or $J = X_s$ the result follows immediately from the construction of the S cut and T cut.) We have the definition

$$\mathcal{Y}_J = \{\text{all subclusters that can be reached from } J \text{ along a directed path in } \mathcal{G}(\mathcal{N},\mathcal{A})\}, \quad (A2)$$

$$\mathcal{X}_I = \mathcal{N} - \mathcal{Y}_J. \quad (A3)$$

Now $I \notin \mathcal{Y}_J$; otherwise $\mathcal{G}(\mathcal{N},\mathcal{A})$ would contain a directed cycle. Therefore, $I \in \mathcal{X}_I$, so $(\mathcal{X}_I, \mathcal{Y}_J)$ is a directed partition in $\mathcal{G}(\mathcal{N},\mathcal{A})$ and (IJ) connects this directed partition. This partition determines a directed partition in G_f^{max} and therefore a minimum cut in $G(N,A)$ according to Proposition 2. Then $(IJ) \subset \mathcal{A}_{mc}$ and because (IJ) is arbitrary $\mathcal{A}_{sg} \subset \mathcal{A}_{mc}$ follows.

Proposition 4. \mathcal{A}_{mc} is the set of arcs of $\mathcal{G}(\mathcal{N},\mathcal{A})$ that will be saturated by all the maximal flows (the ‘‘weak’’ links of the network).

Proof. We have $(ij) \in \mathcal{A}_{mc}$ and G_f^{max} the residual graph for a maximal flow. By the definition of \mathcal{A}_{mc} there exists a minimum cut (X,Y) such that $i \in X$ and $j \in Y$. This minimum cut is a directed cut in G_f^{max} according to Proposition 2, so $r_{ij}=0$, i.e., the arc (ij) is saturated. Conversely, let the arc (ij) with capacity c_{ij} be saturated by all the possible maximal flows through the network, that is, flows with value $f=f_{max}$. Let us assume now that (ij) is not part of any

minimum cut. Therefore, if we decrease the capacity c_{ij} by a small value $\epsilon > 0$, $c_{ij} \rightarrow c_{ij}^* = c_{ij} - \epsilon > 0$, the maximum flow through the network will still have the value f_{max} , according to the Ford-Fulkerson theorem. However, none of the maxi-

mal flows will now “fit” through the network, in particular through the arc (ij) . This is a contradiction; therefore, the arc (ij) must be part of a minimum cut, that is, $(ij) \in \mathcal{A}_{mc}$.

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